2013

Mathmation

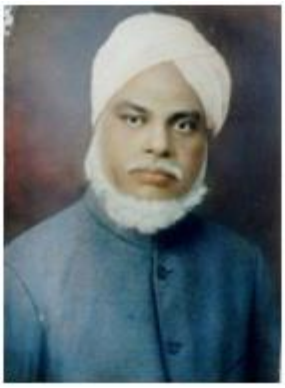
Compendium of Mathematical Information

Students Endaveour
PG & Research Department of Mathematics
Jamal Mohamed College (Autonomous)
College with Potential for Excellence
Accredited with A Grade by NAAC – CGPA 3.6 OUT 4.0
(Affiliated to Bharathidasan University)
Trichy- 20

Jamal Mohamed College (Autonomous)

College with Potential for Excellence
Accredited with A Grade by NAAC – CGPA 3.6 out of 4.0
(Affiliated by Bharathidasan University)
Tiruchirappalli-620020

Founders



Hajee M. Jamal Mohamed

Janab N.M.Khajamian Rowther

Our hearty thanks to Janab M.J. Jamal Mohamed Bilal, President, **Dr. A.K Khaja Nazemudeen,** Secretary and Correspondent, Hajee K.A Khaleel Ahamed, Treasurer, Hajee M.J Jamal Mohamed, Assistant Secretary, Dr.S. Mohamed Salique, Principal, Head and staff members of the Department of Mathematics.

Editorial Board

Introduction

This booklet is the brain child of motivated Maths students & Scholars who wish to disseminate mathematical information regarding the reputed Mathematical Institutions, current events, unsolved problems, Millennium prize problems, puzzles, solutions etc.,

Contents

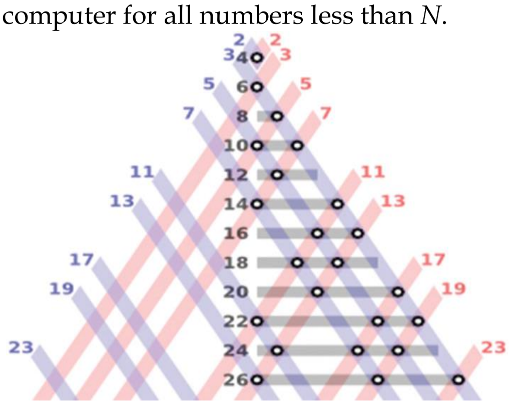
The even integers as sums of two primes

Can every even number greater than 2 be written as the sum of two primes? The affirmative answer to this question is known as Goldbach's conjecture since it was suggested by the Prussian mathematician Christian Goldbach in correspondence with Euler in 1742. (Goldbach's formulation of the conjecture was slightly different because he considered 1 to be a prime number, a convention that was common throughout the 18th and 19th centuries.) This is one of the most famous unsolved problems in mathematics, its acclaim being due in large part to two important characteristics – the problem is easy to state and understand, and it has remained unsolved for a very long time. Incidentally, it is also the subject of a 1992 novel, *Uncle Petros and Goldbach's Conjecture*, by Apostolos Doxiadis, which first appeared in English in 2000 along with an offer from the publishers to award a prize of one million dollars to anyone who could prove the conjecture within two years.

The conjecture has been checked numerically for all even numbers less than 10^{18} and in 1973 Chen Jingrun proved that all sufficiently large even numbers can be written as the sum of a prime and a number with at most two prime factors. It is also known that the number of even numbers which cannot be written as the sum of two primes is very small (it has density zero among the integers). But there is still a chasm to fill between these results to establish the conjecture and Faber and Faber were no doubt pretty confident that their money was safe.

If Goldbach's conjecture is assumed true, then it follows that all odd numbers greater than 5 are the sum of three primes since the odd number can be written as 3 plus an even number greater than 2. This result is known as the ternary Goldbach conjecture, though the term conjecture can now be dropped since it was proved earlier this year. Unfortunately, the ternary result does not imply the original conjecture. If we take an odd number written as the sum of three primes and subtract one of the odd primes from it we will certainly get an expression for an even number as the sum of two primes, but there is no guarantee that all even numbers can be expressed in this way. The ternary result does lead to an improved result on the main conjecture, however, as it can be shown to imply that all even numbers are the sum of at most 4 primes.

The credit for the proof of the ternary conjecture goes to the Peruvian mathematician Harald Helfgott, though as with the resolution of many other long-standing unsolved problems, Helfgott has supplied the final piece in a jigsaw which was begun some time ago and which has involved many great minds. In fact Vinogradov showed in 1937, using a technique called the circle method due to Hardy and Littlewood, that all sufficiently large odd numbers can be written as the sum of three primes, or in other words that there can only be a finite number of odd numbers that cannot be expressed in this way, the only issue since that time has been to derive a number N above which the ternary conjecture holds and which is small enough to allow direct checking via computer for all numbers less than N.



Some key milestones in the story are $N = 10^{7{,}000{,}000}$, a figure arrived at by Vinogradov's student Borozdin in 1956, $N = 10^{1346}$ achieved by Liu Ming-Chit and Wang Tian-Ze in 2002 (it is perhaps worth noting here that the number of atoms in the observableuniverse is estimated to be roughly 10^{80}), and now Helfgott has reduced N to 10^{29}. Working with David Platt from the University of Bristol, Helfgott subsequently confirmed the conjecture numerically for all odd numbers less than this. I am not sure if anyone has checked the calculations, but all the chatter on the internet seems to suggest that there is a high degree of confidence in the result.

Another celebrated unsolved problem in number theory is the twin-prime conjecture which states that there are infinitely many primes p for which $p + 2$ is also prime. The history of this problem is a little harder to trace but it dates at least to the mid-nineteenth century and so it shares with Goldbach's conjecture both the important characteristics of simplicity of expression and longevity (though, as far as I know, no novel). Gaps between prime numbers have long been known to be somewhat slippery animals that give up their secrets begrudgingly, though it is easy to prove that there exist arbitrarily long sequences of composite numbers (just consider $n! + 2$, $n! + 3$,…,

$n! + n$). The proof of the prime number theorem at the end of the nineteenth century which itself was the culmination of almost a century of effort, showed that the 'average' gap between primes near p is roughly $\log p$ so that the gaps get bigger (on average), though rather slowly. For large numbers the average gap between primes is approximately 2.3 times the number of digits.

The prime twin conjecture has not been proved, but some significant progress has been made in 2013. Instead of asking if there are infinitely many primes p for which $p + 2$ is prime, we could ask whether there is a number N such that there are infinitely many prime gaps which are less than or equal to N. This is the bounded gaps conjecture and until 2013 it was unproven, but in May Yitang Zhang from the University of New Hampshire established that such an N does exist. Contrary to the stereotype of new results like this being the domain of brilliant youngsters, Zhang is older than me (there's hope yet!) and (until now) relatively unknown. While Helfgott's work is an impressive refinement of previous work, Zhang's result appears to be entirely new and opens up fresh avenues for research. That is not to say that Zhang did not rely on previous work of others, in particular he built on the important contributions made by Goldston, Pintz and Yıldırım, who in 2005 proved a result which showed that there will always be pairs of primes much closer together (in a sense which can be made precise) than the average spacing predicts but who couldn't bound the gaps by any finite number.

Zhang's work was quickly hailed as a breakthrough and number theorists immediately got to work seeing if they could improve the result by reducing N. Zhang proved the bounded gaps conjecture for N = 70,000,000, the twin primes conjecture corresponds to $N = 2$. With appropriate coordination (as supplied in this case by Terence Tao) the internet allows rapid exchange of ideas and results and the value of N has been steadily reduced during the last few months with improvements announced almost daily. As of 17 August, N had been reduced to 4680, all based on refinements of Zhang's analysis.

The publication of each of these results was pretty momentous and the fact that they both occurred in the same month only served to heighten the excitement within the mathematical fraternity. They also shed light on some changes in the way mathematics research is conducted nowadays. In the first case, we see a problem succumbing to a combined assault from theory and number-crunching power. This is of course not new, the four colour theorem was solved in this way back in 1976, but the inexorable increase in computing power will no doubt make many more problems amenable to this type of approach. In the second case we see the power of the internet at work. The speed with which number theorists have reduced Zhang's value of N strikes me as nothing short of incredible. What would have in the past taken many years has been compressed into months simply by providing an appropriate vehicle for communal research (Tim Gowers' polymath project, begun in 2009).

Both examples also serve to illustrate the general pattern that can often be observed in the resolution of really hard mathematical problems. A breakthrough is achieved which is quickly refined, refinements getting progressively less significant, until a hiatus is reached awaiting the next big breakthrough. The proof of Fermat's Last Theorem followed a similar pattern – Andrew Wiles providing the final breakthrough and Richard Taylor subsequently helping to correct an important technical flaw. The word on the street, by which I mean what I read on Terence Tao's blog, is that Zhang's breakthrough will not be sufficient to lead to a proof of the prime-twin conjecture. The methods may be amenable to further refinement but it is unrealistic to suppose that they will allow mathematicians to reach $N = 2$. So another breakthrough will be required, maybe more than one, but there appears to be a general acceptance that we are considerably closer to a proof than we were a year ago.

Courtesy: Mathematics TODAY OCTOBER 2013

The Beauty of Bounded Gaps

Yitang Zhang, lecturer in mathematics at the University of New Hampshire

A huge discovery about prime numbers—and what it means for the future of math.

Yitang "Tom" Zhang, a popular math professor at the University of New Hampshire, stunned the world of pure mathematics when he announced that he had proven the "bounded gaps" conjecture about the distribution of prime numbers—a crucial milestone on the way to the even more elusive twin primes conjecture, and a major achievement in itself.

The stereotype, outmoded though it is, is that new mathematical discoveries emerge from the minds of dewy young geniuses. But Zhang is over 50. What's more, he hasn't published a paper since 2001. Some of the world's most prominent number theorists have been hammering on the bounded gaps problem for decades now, so the sudden resolution of the problem by a seemingly inactive mathematician far from the action at Harvard, Princeton, and Stanford came as a tremendous surprise.

But the fact that the conjecture is *true* was no surprise at all. Mathematicians have a reputation of being no-bullshit hard cases who don't believe a thing until it's locked down and proved. That's not quite true. All of us believed the bounded gaps conjecture before Zhang's big reveal, and we all believe the twin primes conjecture even though it remains unproven. Why?

Let's start with what the conjectures say. The prime numbers are those numbers greater than 1 that aren't multiples of any number smaller than themselves and greater than 1; so 7 is a prime, but 9 is not, because it's divisible by 3. The first few primes are: 2, 3, 5, 7, 11, 13 …

Every positive number can be expressed in just one way as a product of prime numbers. For instance, 60 is made up of two 2s, one 3, and one 5. (This is why we don't take 1 to be a prime, though some mathematicians have done so in the past; it breaks the uniqueness, because if 1 counts as prime, 60 could be written as 2 x 2 x 3 x 5 and 1 x 2 x 2 x 3 x 5 and 1 x 1 x 2 x 2 x 3 x 5 ...)

The primes are the atoms of number theory, the basic indivisible entities of which all numbers are made. As such, they've been the object of intense study ever since number theory started. One of the very first theorems in number theory is that of Euclid, which tells us that the primes are infinite in number; we will never run out, no matter how far along the number line we let our minds range.

But mathematicians are greedy types, not inclined to be satisfied with mere assertion of infinitude. After all, there's infinite and then there's *infinite*. There are infinitely many powers of 2, but they're very rare. Among the first 1,000 numbers, there are only 10 powers of 2: 1, 2, 4, 8, 16, 32, 64, 128, 256, and 512.

There are infinitely many even numbers, too, but they're much more common: exactly 500 out of the first 1,000. In fact, it's pretty apparent that out of the first X numbers, just about (1/2)X will be even.

Primes, it turns out, are intermediate—more common than the powers of 2 but rarer than even numbers. Among the first X numbers, about X/log(X) are prime; this is the Prime Number Theorem, proven at the end of the 19^{th} century by Hadamard and de la Vallée Poussin. This means, in particular, that prime numbers get less and less common as the numbers get bigger, though the decrease is very slow; a random number with 20 digits is half as likely to be prime as a random number with 10 digits.

Naturally, one imagines that the more common a certain type of number, the smaller the gaps between instances of that type of number. If you're looking at an even number, you never have to travel farther than 2 numbers forward to encounter the next even; in fact, the gaps between the even numbers are always *exactly* of size 2. For the powers of 2, it's a different story. The gaps between successive powers of 2 grow exponentially, and there are finitely many gaps of any given size; once you get past 16, for instance, you will never again see two powers of 2 separated by a gap of size 15 or less.

Those two problems are easy, but the question of gaps between consecutive primes is harder. It's so hard that, even after Zhang's breakthrough, it remains a mystery in many respects.

And yet we think we know what to expect, thanks to a remarkably fruitful point of view—we think of primes as *random numbers*. The reason the fruitfulness of this viewpoint is so remarkable is that the viewpoint is so very, very false. Primes are not random! Nothing about them is arbitrary or subject to chance. Quite the opposite—we take them as immutable features of the universe, and carve them on the golden records we shoot out into interstellar space to prove to the ETs that we're no dopes.

If you start thinking really hard about what "random" *really* means, first you get a little nauseated, and a little after that you find you're doing analytic philosophy. So let's not go down that road.

Instead, take the mathematician's path. The primes are not random, but it turns out that in many ways they *act as if they were*. For example, when you divide a random number by 3, the remainder is either 0, 1, or 2, and each case arises equally often. When you divide a big prime number by 3, the quotient can't come out even; otherwise, the so-called prime would be divisible by 3, which would mean it wasn't really a prime at all. But an old theorem of Dirichlet tells us that remainder 1 shows up about equally often as remainder 2, just as is the case for random numbers. So as far as "remainder modulo 3" goes, prime numbers, apart from not being multiples of 3, look random.

What about the gaps between consecutive primes? You might think that, because prime numbers get rarer and rarer as numbers get bigger, that they also get farther and farther apart. On average, that's indeed the case. But what Yitang Zhang just proved is that there are infinitely many pairs of primes that differ by at most 70,000,000. In other words, that the gap between one prime and the next is bounded by 70,000,000 infinitely often—thus, the "bounded gaps" conjecture.

On first glance, this might seem a miraculous phenomenon. If the primes are tending to be farther and farther apart, what's causing there to be so many pairs that are close together? Is it some kind of prime gravity?

Nothing of the kind. If you strew numbers at random, it's very likely that some pairs will, by chance, land very close together. (The left-hand picture on this page is a nice illustration of how this works in the plane; the points are chosen independently and completely randomly, but you see some clumps and clusters all the same.)

It's not hard to compute that, if prime numbers behaved like random numbers, you'd see precisely the behavior that Zhang demonstrated. Even more: You'd expect to see infinitely many pairs of primes that are separated by only 2, as the twin primes conjecture claims.

(The one computation in this article follows. If you're not onboard, avert your eyes and rejoin the text where it says "And a lot of twin primes …")

Among the first N numbers, about N/log N of them are primes. If these were distributed randomly, each number n would have a 1/log N chance of being prime. The chance that n and n+2 are *both* prime should thus be about (1/log N)^2. So how many pairs of primes separated by 2 should we expect to see? There are about N pairs (n, n+2) in the range of interest, and each one has a (1/log N)^2 chance of being a twin prime, so one should expect to find about N/(log N)^2 twin primes in the interval.

There are some deviations from pure randomness whose small effects number theorists know how to handle; a more refined analysis taking these into account suggests that the number of twin primes should in fact be about 32 percent greater than N/(log N)^2. This better approximation gives a prediction that the number of twin primes less than a quadrillion should be about 1.1 trillion; the actual figure is 1,177,209,242,304. That's a lot of twin primes.

And a lot of twin primes is exactly what number theorists expect to find no matter how big the numbers get—not because we think there's a deep, miraculous structure hidden in the primes, but *precisely because we don't think so*. We expect the primes to be tossed around at random like dirt. If the twin primes conjecture were false, *that* would be a miracle, requiring that some hitherto unknown force be pushing the primes apart.

Not to pull back the curtain too much, but a lot of famous conjectures in number theory are like this. The Goldbach conjecture that every even number is the sum of two primes? The ABC conjecture, for which Shin Mochizuki controversially claimed a proof last fall? The conjecture that the primes contain arbitrarily long arithmetic progressions, whose resolution by Ben Green and Terry Tao in 2004 helped win Tao a Fields Medal? All are immensely difficult, but they are all exactly what one is guided to believe by the example of random numbers.

It's one thing to know what to expect and quite another to prove one's expectation is correct.

Despite the apparent simplicity of the bounded gaps conjecture, Zhang's proof requires some of the deepest theorems of modern mathematics, like Pierre Deligne's results relating averages of number-theoretic functions with the geometry of high-dimensional spaces. (More classically minded analytic number theorists are already wondering whether Zhang's proof can be modified to avoid such abstruse stuff.)

Building on the work of many predecessors, Zhang is able to show in a rather precise sense that the prime numbers look random in the first way we mentioned, concerning the remainders obtained after division by many different integers. From this (following a path laid out by Goldston, Pintz, and Yıldırım, the last people to make any progress on prime gaps) he can show that the prime numbers look random in a totally different sense, having to do with the sizes of the gaps between them. Random is random!

Zhang's success (along with the work of Green and Tao) points to a prospect even more exciting than any individual result about primes—that we might, in the end, be on our way to developing a richer theory of randomness. How wonderfully paradoxical: What helps us break down the final mysteries about prime numbers may be new mathematical ideas that structure the concept of structurelessness itself.

(A few suggestions for further reading for those with more technical tastes: Number theorist Emmanuel Kowalski offers a first report on Zhang's paper. And here's Terry Tao on the dichotomy between structure and randomness.)

http://www.slate.com/articles/health_and_science/do_the_math/2013/05/yitang_zhang_twin_primes_conjecture_a_huge_discovery_about_prime_numbers.2.html

Angel Problem

"The angel problem is a question in game theory proposed by John Horton Conway.[1] The game is commonly referred to as the Angels and Devils game. The game is played by two players called the angel and the devil. It is played on an infinite chessboard (or equivalently the points of a 2D lattice). The angel has a power k (a natural number 1 or higher), specified before the games starts. The board starts empty with the angel at the origin. On each turn, the angel jumps to a different empty square which could be reached by at most k moves of a chess king. i.e. The distance from the starting square is at most k in the infinity norm.) The devil, on his turn, may add a block on any single square not containing the angel. The angel may leap over blocked squares, but cannot land on them. The devil wins if the angel is unable to move. The angel wins by surviving indefinitely."

"The angel problem is: can an angel with high enough power win?"

"There must exist a winning strategy for one of the players. If the devil can force a win then he can do so in a finite number of moves. If the devil cannot force a win then there is always an action that the angel can take to avoid losing and a winning strategy for her is always to pick such a move. More abstractly, the "pay-off set" (i.e., the set of all plays in which the angel wins) is a closed set (in the natural topology on the set of all plays), and it is known that such games are determined."

"Conway offered a reward for a general solution to this problem ($100 for a winning strategy for an angel of sufficiently high power, and $1000 for a proof that the devil can win irrespective of the angel's power). Progress was made first in higher dimensions, with some beautiful proofs. In late 2006, the original problem was solved when independent proofs appeared, showing that an angel can win. Bowditch proved that a 4-angel can win[2] and Máthé[3] and Kloster[4] gave proofs that a 2-angel can win."

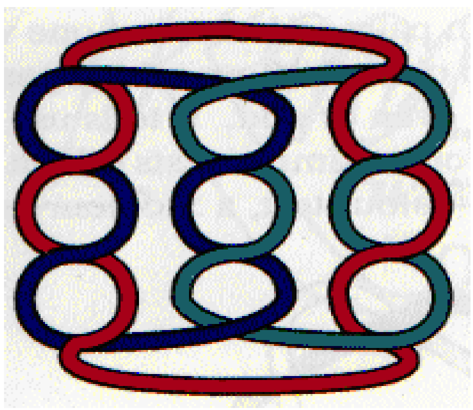
http://en.wikipedia.org/wiki/Angel_problem

Links and Knots

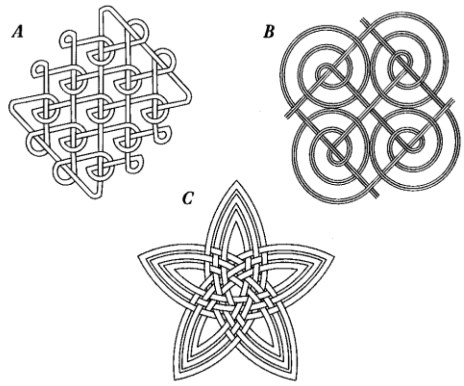
Heather McLeay

These simple puzzles have been selected because they require no knowledge of knot theory, just a careful inspection of the patterns. The puzzles are taken from The Knots Puzzle Book by Heather McLeay published by Tarquin Publications [1]. The book gives a simple introduction to the classification of knots and a little about prime knots, crossing numbers and knot arithmetic, just enough theory for solving some more puzzles. There is serious mathematics at the next stage and you might like to go to the Knots Exhibition website to find answers to questions such as "What sort of mathematical theory begins with the simple reef knot?" and "Why are mathematicians interested in such problems?" See our review of this exhibition.

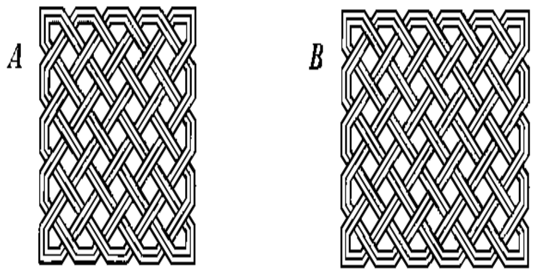
A mathematical knot is really an "endless knot" made from a single strand, with crossovers where one piece of the rope crosses another piece, and the ends joined together to make it continuous. The rope can then be rearranged, but not untied, and however it is handled the knot remains essentially the same. Knots which can be rearranged to have no crossovers at all are called unknots.



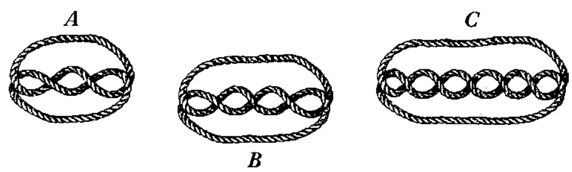
Some of these decorative knot patterns are formed from a single strand, others from several strands. How many separate strands are needed to make up each of these designs?



The two Celtic knot patterns below look very similar except that one might be described as a 'four by four' and the other as a 'five by four'. What is the fundamental difference between them? Some patterns like this can be formed from a single strand and others need several strands, what can you say about the dimensions of these two types?

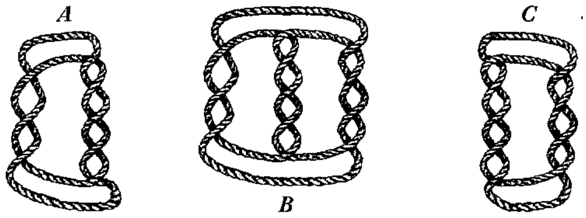


Which of these diagrams show knots made from single strands and which are made from linked strands?



How many linked strands do you need to make a ten crossing version of the diagram above?

Here are some examples of more complicated links. How many separate strands are there in each case?



Is there an easy way to predict how many strands knots like these will have?

In his article entitled Celtic Design, in the March 1998 issue of Mathematics in School [2], John Matthews describes how he used the !Draw package on an Acorn 310 to design five basic squares and then to 'grid lock', 'copy' and 'transform' the squares, assembling the copies to produce Celtic patterns and to design his own. You might like to try this for yourself. In the same article John produces sets of tiles, with just the right number of each of the five basic squares in the set, so that the tiles can be fitted together, like a jigsaw, to make a given design. The article gives sets of tiles for five designs. With this idea you can easily produce lots of the tiles and make up your own designs. Here we are at the borders of mathematics, art and design. Thank you John for a delightful starter to many more knotty possibilities.

[1] *Address for Tarquin Publications: Stradbroke, Diss, Norfolk IP21 5JP, England.*
[2] *For further information you can write to John Matthews, Auchenharvie Academy, Saltoats Rd, Stevenston, Scotland, KA20 3JW.*

Another Way of Thinking: A Review of Mathematical Models of Crime

With all the numbers bandied about in the media, one might be forgiven for thinking that crime rates are up – and not to any good either. Homicide rates, conviction rates, the increase in robberies with accompanying charts figures galore. Statistics has generally gone hand in hand with crime and all this number crunching has traditionally left a bitter taste in the public's mouth. Add mathematics to this mix, and this bitter taste turns noxious.

Though the popularity of television shows like *NUMB3RS* has made mathematics more palatable to the crime-fearing public, mathematics is still viewed as a distant, fearsome relative of criminology. This, despite its long association with the social sciences. The use of mathematics to describe social phenomena has its roots in the 'Era of Enlightenment' in the nineteenth century with the birth of statistics and probability theory and their use in the collection and analysis of social statistics. However, this collaboration was short-lived as the social sciences turned away from statistics and moved towards a psychological approach to understanding behavior.

Criminal behaviour and activities have evolved in tandem with changes in technology. Criminal activities now include arms, drugs and human trafficking, money laundering, cyber-crime, identity theft and gangs with international links. Crime has become more sophisticated, organised and transnational. With the changing nature of crime, traditional approaches to tack-ling it are fast becoming obsolete and there is a growing need for a new way of thinking to meet this challenge head on.

Mathematical modeling and numerical simulation partners in crime

The advent of cheaper, more powerful computers has ushered in a revolution in mathematics. Mathematical modeling uses math-ematics to transform real-world systems into abstract models so as to understand simulate or make predictions about their behaviour. Some of these systems may have no analytical solutions and numerical simulation via the computer must be used to determine their approximate solutions. These solutions may be in the form of graphs showing the system's behaviour over time as well as its sensitivity to variations in key model parameters.

Mathematical modeling and numerical simulation of systems have once again resulted in a friendship of sorts between mathematics and criminology. When modeling criminal behaviour and crime, since human behaviour is inherently nonlinear, we assume that they may best be described by a nonlinear system. This is contrary to the models generally used by policy-makers, who are susceptible to what Ball [1] calls linear thinking. This has led to the development of linear models of human behaviour with their inherent assumptions that cause-and-effect relationships are identifiable and that there is proportionality between inputs and outputs. These properties make linear systems particularly useful for prediction and manipulation hence their popularity in modelling.

However, when a system contains nonlinear terms, analytical solutions may be difficult to obtain so that numerical methods must be used to obtain the approximate solutions. In nonlinear systems, proportionality does not hold and there is a disproportional relationship between cause and effect: small changes in key parameters can trigger large changes in crime rate. Another feature of nonlinearity is the existence of bifurcation points in key parameters. The bifurcations give 'tipping points' of the system at which the system may make a sudden transition to a new, very different behaviour.

'Mathematical modelling and numerical simulation complement the traditional empirical and experimental approaches to research' [3]. Modelling is especially important in criminology since it helps organise and visualise existing data, identify areas with missing data and is relatively inexpensive and more practical than carrying out an actual experiment. Modelling also offers a means of varying conditions so as to conduct social experiments but without the ethics and costs attached to experimenting on human beings. The insights provided by the models may be especially helpful to those in authority who are charged with the responsibility of designing policies often with a lack of available data.

Crime 'math'ers

Terminology used to characterise crime and criminal behaviour seems to lend itself almost naturally to the use of existing mathematical models so as to mathematically represent a particular crime situation. Some of these terms include crime waves, the spread of crime, crime epidemic, the migration of criminals, criminals preying upon the population and the formation of crime hotspots. Two mathematical approaches used in the modelling of crime and criminal behaviour are described next.

Modelling via differential equations

In nature, transport occurs in fluids through the combination of advection and diffusion. Reaction diffusion advection systems are used to study the spread of wave like behavior in a number of fields such as the migration of invasive species, the propagation of genes and the spread of chemical reactions. A reaction diffusion advection model has been applied by Stanford researchers Nancy Rodríguez, Henri Berestycki and Lenya Ryzhik to describe and reduce the spread of crime waves outward from crime hotspots. Reaction diffusion partial differential equations have also been used to study the formation, dynamics, and steady state properties of crime hotspots and to explain why these hotspots may either be displaced or eradicated by police action.

Criminal behavior and violence may also be treated as a socially infectious disease using concepts borrowed from epidemiology. Researchers have recognized the propensity for violent acts to cluster, to spread from one area to another and to mutate and have suggested applying existing techniques from mathematical epidemiology to treat the spread of violence in a population.

In one of the earliest papers to acknowledge the social nature of crime, individual crime was treated as a function of exposure to crime prone peers, where an individual is influenced in his choice to commit a criminal act by his perceived probability of punishment as obtained from his acquaintances. Ormerod, Mounfield and Smith applied an infectious disease model consisting of a system of coupled, non linear ordinary differential equations to violent crime and burglary in the UK. The model divided the population into four groups three dependent on their susceptibility to commit crimes and one group representing those in jail. The model was used to test the effect of crime fighting policies on the criminal population.

A similar model was developed for the growth of gangs in a population by dividing the population into four distinct groups based on gang status and risk factors with respect to gang membership. The model examined the impact of various crime fighting strategies by changing parameter values like imprisonment and recidivism rates and identified bifurcation points which resulted in the disappearance of gang members from the population.

Closely related to infectious disease models are predator prey models which also use systems of ordinary differential equations and seem to be a natural fit for modeling criminals who 'prey upon' the public. These models have also been used in the inverse setting to describe the interactions between policemen (predators) and criminals (prey) and to examine the effects of changes in policy and law enforcement. Other models include Nuno et al. Who modeled a dynamical system based on routine activity theory containing a group of motivated off Enders Y, suitable targets X and a lack of guardianship. The model consisted of owners X who are the prey, criminals Y who are the predators of X, and security guards Z who are predators of both X and Y. Nuno et al. also compared two different strategies (upgrading police forces and increasing social measures) for fighting crime in a criminal prone self protected society divided in to n different socio economic classes. Criminals preying upon the villagers who banded together in group defense were modeled so that the criminals switched between areas targeting the less populated areas. Police efforts to catch criminals were included in the model by applying constant effort and constant yield harvesting functions to capture the criminals.

Agent based models

Another approach to modeling crime and criminal behavior in which numerical simulation plays an important role uses agent based models (ABM). These are made up of a collection of autonomous decision making entities called agents who interact with each other and their environment, according to a set of specified behavioral rules.

When used to model crime, the agents generally represent people criminals, potential victims, police etc. The agents populate an artificial environment that is designed to reflect features such as buildings, a street network, a social network, or barriers to movement etc. The movement and interaction of agents are defined by either equations or rules. The inherently spatial nature of human movement, interactions and the role of place in influencing these interactions are naturally incorporated into these models.

In crime, agent based models are popular in investigating the environmental aspects of criminal behavior like the mapping of crime hotspots and crime displacement, street gang rivalries, street robberies and burglary. Agent based models have also been combined with Geographic Information Systems (GIS) to simulate dynamic spatial systems. Other applications include the dependence of the frequency of violence and criminal activity on population size and whether a society without crime is possible. While the previous modelling approach was characterized by a 'top-down' modelling approach where the behavior of the system is described at the start by a system of equations, the agent-based model is characterized by a 'bottom-up' approach where there is emergent behavior.

Towards another way of thinking

Models may be used to guide decision-making, develop policies or to evaluate specific strategies aimed at reducing crime. In developing models, the question of model validity or how well the model rep-resents the real-world situation for which it is designed naturally arises. Model validation techniques include consultation with experts about the model design and its behavior, parameter variability-sensitivity analyses of model behavior and the use of statistical tests and procedures to compare model output for different experimental conditions with experimental data.

Experimental data in this research refers to crime data and statistics. In designing models of crime, there are challenges associated with the data collection process. Some of these challenges include case attrition where cases that enter the system get lost somewhere along the way, lack of data on offenders, lack of self-report studies, unreported crime due to a lack of trust in the police and desensitization to crime which may result in varying degrees of tolerance to crime. This has led to concerns about whether crime data should be viewed as representative of the crime situation in a particular area and may lead to invalid explanations of crime phenomenon and ineffectual policies to re-duce crime. Thus, most of the models reviewed were used not to predict future trends but for insight into the behavior of the system.

In all of the models reviewed, we noted that building a crime model involved a multidisciplinary approach so as to bridge the gap between the physical and the social sciences. The 'ideal type of the division of labour in quantitative social science would be one where the sociologist formulates a theory, the mathematician translates it into a mathematical model, and the statistician pro-vides the tool for estimating the model'.

Courtesy: Mathematics TODAY OCTOBER 2013

Millennium Prize Problems

The Millennium Prize Problems are seven problems in mathematics that were stated by the Clay Mathematics Institute in 2000. As of July 2013, six of the problems remain unsolved. A correct solution to any of the problems results in a US$1,000,000 prize (sometimes called a *Millennium Prize*) being awarded by the institute. The Poincaré conjecture, the only Millennium Prize Problem to be solved so far, was solved by Grigori Perelman, but he declined the award in 2010.

Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs. NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

Unsolved problems in mathematics

Additive number theory

Beal's conjecture

Beal's conjecture is a conjecture in number theory:

If $A^x + B^y = C^z$, where *A*, *B*, *C*, *x*, *y*, and *z* are positive integers with *x*, *y*, *z* > 2, then *A*, *B*, and *C* have a common prime factor.

Billionaire banker Andrew Beal formulated this conjecture in 1993 while investigating generalizations of Fermat's last theorem.[1] It has been claimed that the same conjecture was independently formulated by Robert Tijdeman and Don Zagier, and it has also been referred to as the Tijdeman Zagier conjecture.

For a proof or counterexample published in a refereed journal, Beal initially offered a prize of US $5,000 in 1997, raising it to $50,000 over ten years,[4] but has since raised it to US $1,000,000

Gilbreath's conjecture

Gilbreath's conjecture is a hypothesis, or a conjecture, in number theory regarding the sequences generated by applying the forward difference operator to consecutive prime numbers and leaving the results unsigned, and then repeating this process on consecutive terms in the resulting sequence, and so forth. The statement is named after mathematician Norman L. Gilbreath who, in 1958, presented it to the mathematical community after observing the pattern by chance while doing arithmetic on a napkin.[1] In 1878, eighty years before Gilbreath's discovery, François Proth had, however, published the same observations along with an attempted proof, which was later shown to be false.

Algebra

Hilbert's sixteenth problem

Hilbert's 16th problem was posed by David Hilbert at the Paris conference of the International Congress of Mathematicians in 1900, as part of his list of 23 problems in mathematics.

The original problem was posed as the *Problem of the topology of algebraic curves and surfaces* (*Problem der Topologie algebraischer Kurven und Flächen*).

Actually the problem consists of two similar problems in different branches of mathematics:

- An investigation of the relative positions of the branches of real algebraic curves of degree *n* (and similarly for algebraic surfaces).
- The determination of the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree *n* and an investigation of their relative positions.

The first problem is yet unsolved for *n* = 8. Therefore, this problem is what usually is meant when talking about Hilbert's sixteenth problem in real algebraic geometry. The second problem also remains unsolved: no upper bound for the number of limit cycles is known for any *n*>1, and this is what usually is meant by Hilbert's sixteenth problem in the field of dynamical systems.

Algebraic geometry

Jacobian conjecture

In mathematics, the Jacobian conjecture is a celebrated problem on polynomials in several variables. It was first posed in 1939 by Ott-Heinrich Keller. It was later named and widely publicised by Shreeram Abhyankar, as an example of a question in the area of algebraic geometry that requires little beyond knowledge of calculus to state.

The Jacobian conjecture is notorious for the large number of attempted proofs that turned out to contain subtle errors. As of February 2013, there are no plausible claims to have proved it.

Nakai conjecture

In mathematics, the Nakai conjecture is an unproven characterization of smooth algebraic varieties, conjectured by Japanese mathematician Yoshikazu Nakai in 1961. It states that if *V* is a complex algebraic variety, such that its ring of differential operators is generated by the derivations it contains, then *V* is a smooth variety. The converse statement, that smooth algebraic varieties have rings of differential operators that are generated by their derivations, is a result of Alexander Grothendieck.

The Nakai conjecture is known to be true for algebraic curves. A proof of the conjecture would also prove the Zariski-Lipman conjecture, for a complex variety *V* with coordinate ring *R*.

This conjecture states that if the derivations of R are a free module over R, then V is smooth.

Algebraic number theory

Brumer–Stark conjecture

The Brumer–Stark conjecture is a conjecture in algebraic number theory giving a rough generalization of both the analytic class number formula for Dedekind zeta functions, and also of Stickelberger's theorem about the factorization of Gauss sums. It is named after Armand Brumer and Harold Stark.

It arises as a special case (abelian and first-order) of Stark's conjecture, when the place that splits completely in the extension is finite. There are very few cases where the conjecture is known to be valid. Its importance arises, for instance, from its connection with Hilbert's twelfth problem.

Combinatorics

lonely runner conjecture

In number theory, and especially the study of diophantine approximation, the lonely runner conjecture is a conjecture originally due to J. M. Wills in 1967. Applications of the conjecture are widespread in mathematics; they include view obstruction problems[1] and calculating the chromatic number of distance graphs and circulant graphs.[2] The conjecture was given its picturesque name by L. Goddyn in 1998

Singmaster's conjecture

Singmaster's conjecture is a conjecture in combinatorial number theory in mathematics, named after the British professor David Singmaster who proposed it in 1971. It says that there is a finite upper bound on the multiplicities of entries in Pascal's triangle (other than the number 1, which appears infinitely many times). It is clear that the only number that appears infinitely many times in Pascal's triangle is 1, because any other number x can appear only within the first $x + 1$ rows of the triangle. Paul Erdős said that Singmaster's conjecture is probably true but he suspected it would be very hard to prove.

Let $N(a)$ be the number of times the number $a > 1$ appears in Pascal's triangle. In big O notation, the conjecture is:

$$N(a) = O(1).$$

Graph theory

Total coloring

In graph theory, total coloring is a type of graph coloring on the vertices and edges of a graph. When used without any qualification, a total coloring is always assumed to be *proper* in the sense that no adjacent edges and no edge and its end vertices are assigned the same color. The total chromatic number $\chi''(G)$ of a graph G is the least number of colors needed in any total coloring of G.

Hadwiger conjecture

In graph theory, the Hadwiger conjecture (or Hadwiger's conjecture) states that, if all proper colorings of an undirected graph G use k or more colors, then one can find k disjoint connected subgraphs of G such that each subgraph is connected by an edge to each other subgraph. Contracting the edges within each of these subgraphs so that each subgraph collapses to a single supervertex produces a complete graph K_k on k vertices as a minor of G.

This conjecture, a far-reaching generalization of the four-color problem, was made by Hugo Hadwiger in 1943 and is still unsolved. Bollobás, Catlin & Erdős (1980) call it "one of the deepest unsolved problems in graph theory

Number theory (prime numbers)

Landau's problems

At the 1912 International Congress of Mathematicians, Edmund Landau listed four basic problems about primes. These problems were characterised in his speech as "unattackable at the present state of science" and are now known as Landau's problems. They are as follows:

1. Goldbach's conjecture: Can every even integer greater than 2 be written as the sum of two primes?
2. Twin prime conjecture: Are there infinitely many primes p such that $p + 2$ is prime?
3. Legendre's conjecture: Does there always exist at least one prime between consecutive perfect squares?
4. Are there infinitely many primes p such that $p - 1$ is a perfect square? In other words: Are there infinitely many primes of the form $n^2 + 1$? (sequence A002496 in OEIS).

As of 2013, all four problems are unresolved.

Problems solved recently

Serre's modularity conjecture

In mathematics, Serre's modularity conjecture, introduced by Serre (1975, 1987) based on some 1973–1974 correspondence with John Tate, states that an odd irreducible two-dimensional Galois representation over a finite field arises from a modular form, and a stronger version of his conjecture specifies the weight and level of the modular form. It was proved by Chandrashekhar Khare in the level 1 case[1] in 2005 and later in 2008 a proof of the full conjecture was worked out jointly by Chandrashekhar Khare and Jean-Pierre Wintenberger

Road coloring problem

In graph theory the **road coloring theorem**, known until recently as the **road coloring conjecture**, deals with synchronized instructions. The issue involves whether by using such instructions, one can reach or locate an object or destination from any other point within a network (which might be a representation of city streets or a maze).[1] In the real world, this phenomenon would be as if you called a friend to ask for directions to his house, and he gave you a set of directions that worked no matter where you started from. This theorem also has implications in symbolic dynamics.

The theorem was first conjectured by Roy Adler and Benjamin Weiss (1970). It was proved by Avraham Trahtman (2009)

Hirsch conjecture

In mathematical programming and polyhedral combinatorics, the Hirsch conjecture is the generally false statement that the edge-vertex graph of an n-facet polytope in d-dimensional Euclidean space has diameter no more than $n - d$. That is, any two vertices of the polytope must be connected to each other by a path of length at most $n - d$. The conjecture was first put forth in a letter by Warren M. Hirsch to George B. Dantzig in 1957[1][2] and was motivated by the analysis of the simplex method in linear programming, as the diameter of a polytope provides a lower bound on the number of steps needed by the simplex method.

The Hirsch conjecture was proven for $d < 4$ and for various special cases.[3] The best known upper bounds showed only that polytopes have sub-exponential diameter as a function of n and d.[4] After more than fifty years, a counter-example was announced in May 2010 by Francisco Santos, from the University of Cantabria.[5][6][7] The result was presented at the conference *100 Years in Seattle: the mathematics of Klee and Grünbaum* and appeared in Annals of Mathematics.[8] Specifically, the paper presented a 43-dimensional polytope of 86 facets with a diameter of more than 43. The counterexample has no direct consequences for the analysis of the simplex method, as it does not rule out the possibility of a larger but still linear or polynomial number of steps.

Various equivalent formulations of the problem had been given, such as the d-step conjecture, which states that the diameter of any $2d$-facet polytope in d-dimensional Euclidean space is no more than d.

Angel problem

The angel problem is a question in game theory proposed by John Horton Conway.[1] The game is commonly referred to as the Angels and Devils game. The game is played by two players called the angel and the devil. It is played on an infinite chessboard (or equivalently the points of a 2D lattice). The angel has a power k (a natural number 1 or higher), specified before the game starts. The board starts empty with the angel at the origin. On each turn, the angel jumps to a different empty square which could be reached by at most k moves of a chess king, i.e. the distance from the starting square is at most k in the infinity norm. The devil, on its turn, may add a block on any single square not containing the angel. The angel may leap over blocked squares, but cannot land on them. The devil wins if the angel is unable to move. The angel wins by surviving indefinitely.

The angel problem is: can an angel with high enough power win?

There must exist a winning strategy for one of the players. If the devil can force a win then it can do so in a finite number of moves. If the devil cannot force a win then there is always an action that the angel can take to avoid losing and a winning strategy for it is always to pick such a move. More abstractly, the "pay-off set" (i.e., the set of all plays in which the angel wins) is a closed set (in the natural topology on the set of all plays), and it is known that such games are determined.

Conway offered a reward for a general solution to this problem ($100 for a winning strategy for an angel of sufficiently high power, and $1000 for a proof that the devil can win irrespective of the angel's power). Progress was made first in higher dimensions. In late 2006, the original problem was solved when independent proofs appeared, showing that an angel can win. Bowditch proved that a 4-angel can win[2] and Máthé[3] and Kloster[4] gave proofs that a 2-angel can win. At this stage, it has not been confirmed by Conway who is to be the recipient of his prize offer, or whether each published and subsequent solution will also earn $100 US.

Mathematics Research institutes in India

Chennai Mathematical Institute

Chennai Mathematical Institute is a centre of excellence for teaching and research in the mathematical sciences. It was founded in 1989 as a part of the SPIC Science Foundation, funded by the SPIC group in Chennai. Since 1996, it has been an autonomous institution.

The research groups in Mathematics and Computer Science at CMI are among the best known in the country. Recently, a research group has also been set up in Physics. The Institute has nurtured an impressive collection of PhD students.

The main areas of research in Mathematics pursued at the Institute are algebra, analysis, differential equations, geometry and topology. In Computer Science, the main areas of research are formal methods in the specification and verification of software systems, design and analysis of algorithms, computational complexity theory and computer security. In Physics, research is being carried out mainly in quantum field theory, mathematical physics and string theory.

Harish-Chandra Research Institute

The Harish-Chandra Research Institute (HRI) is an institution dedicated to research in mathematics, and in theoretical physics. It is located in Allahabad, India, and is funded by the Department of Atomic Energy, Government of India.

Research at HRI is focussed on Mathematics and Theoretical Physics. The academic community at HRI consists of faculty members, post-doctoral fellows, and graduate students.

The Institute has a graduate programme leading to the Ph.D. degree. Degrees for the programme are awarded by the Homi Bhabha National Institute. Admissions to the graduate program take place through a Joint Entrance Screening Test, which is organized in collaboration with several other institutions, and an interview. Further, the Institute offers post-doctoral fellowships, and visiting positions at various levels.

Institute of Mathematical Sciences

The Institute of Mathematical Sciences (IMSc), founded in 1962 and based in the verdant surroundings of the CIT campus in Chennai, is a national institution which promotes fundamental research in frontier disciplines of the mathematical and physical sciences: Theoretical Computer Science, Mathematics, Theoretical Physics, as well as in many interdisciplinary fields. Around a hundred researchers, including faculty members, post-doctoral fellows and graduate students, are members of the Institute at any given time, in addition to a large number of visitors from all over the world.

Tata Institute of Fundamental Research

The Tata Institute of Fundamental Research is a National Centre of the Government of India, under the umbrella of the Department of Atomic Energy, as well as a deemed University awarding degrees for master's and doctoral programs. At TIFR, we carry out basic research in physics, chemistry, biology, mathematics, computer science and science education.

Kerala School of Mathematics

Kerala School of Mathematics (KSOM), an institution meant for advanced learning and research in Mathematics, is a joint venture of Kerala State Council for Science, Technology and Environment (KSCSTE), Government of Kerala and Department of Atomic Energy (DAE), Government of India. It is built in the traditional Kerala style architecture and is located in a picturesque setting surrounded by serene hillocks and lavish greenery. It is at a distance of about 15 km. from the seaside city Kozhikode in Malabar region of Kerala, India.